TUTORIAL NOTES FOR MATH4010

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1. DUAL SPACE OF $C([a,b])$

Let us discuss the dual space of $C([a,b])$.

Example 1 **($(C([a,b]))^* = BV([a,b])$).** Every bounded linear functional L on $C([a,b])$ can be represented by a Riemann-Stieltjes integral

$$L(f) = \int_a^b f dg, \quad \forall f \in C([a,b]),$$

where $g \in BV([a,b])$ with $\|L\| = V_a^b(g)$.

Proof. Recall the definition of Riemann-Stieltjes integral. Let x and y be bounded functions on the closed interval $[a,b]$, and

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

a partition of $[a,b]$. The norm of P is defined by

$$\|P\| = \max_{1\leq k\leq n}(t_k - t_{k-1}).$$

We arbitrarily select points $\tau_k \in [t_{k-1}, t_k]$, $1 \leq k \leq n$, and define a Riemann-Stieltjes sum by

$$S(f,g,P,\tau) = \sum_{k=1}^n f(\tau_k)[g(t_k) - g(t_{k-1})].$$

If there is a real number J such that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|J - S(f,g,P,\tau)| < \varepsilon,$$

for every P satisfying $\|P\| < \delta$, then J is called the Riemann-Stieltjes integral of f with respect to g on $[a,b]$, and is denoted by

$$J = \int_a^b f(t)dg(t) = \int_a^b f dg.$$

Recall the definition of $BV([a,b])$. Let g be a real-valued function on the interval $[a,b]$ and let

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

be a partition of $[a,b]$. The variation of g on $[a,b]$ with respect to P is defined by

$$V_a^b(g,P) = \sum_{k=1}^n |g(t_k) - g(t_{k-1})|.$$

If there is a constant M such that

$$V_a^b(g,P) < M,$$

for every partition P of $[a, b]$, then we call g is a function of bounded variation, a BV-function for short, on $[a, b]$ and define the total variation of g over the interval $[a, b]$ as

$$V_a^b(g) = \sup\{V_a^b(g, P) : P \text{ is a partition of } [a, b]\}.$$

The set of all BV-functions on $[a, b]$ is denoted by $BV([a, b])$.

Let $B([a, b])$ be the vector space of all bounded functions on $[a, b]$ endowed with the sup-norm and L a bounded linear functional on $C([a, b])$. Clearly, $C([a, b])$ is a subspace of $B([a, b])$. By the Hahn-Banach theorem, L has an extension $\tilde{L}$ on $B([a, b])$ such that $\|\tilde{L}\|_{(B([a,b]))^*} = \|L\|_{(C([a,b]))^*}$.

To find the function $g \in BV([a, b])$ such that

$$L(f) = \int_a^b f dg, \quad \forall f \in C([a, b]),$$

where $\int_a^b f dg$ is the Riemann-Stieltjes integral. Consider the family of functions in $B([a, b])$ defined by

$$f_t(\tau) = \begin{cases} 1, & a \leq \tau \leq t, \\ 0, & t < \tau \leq b, \end{cases}$$

and define g by

$$g(t) = \begin{cases} 0, & t = a, \\ \tilde{L}(f_t), & a < t \leq b. \end{cases}$$

We claim that $g \in BV([a, b])$. Indeed, let us define

$$\varepsilon_k = sgn(g(t_k) - g(t_{k-1})), \quad 1 \leq k \leq n,$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$. Therefore

$$\begin{aligned}
\sum_{k=1}^n |g(t_k) - g(t_{k-1})| =& \sum_{k=1}^n \varepsilon_k \left[g(t_k) - g(t_{k-1})\right] \\
=& \varepsilon_1 \tilde{L}\left(f_{t_1}\right) + \sum_{k=2}^n \varepsilon_k \left[\tilde{L}\left(f_{t_k}\right) - \tilde{L}\left(f_{t_{k-1}}\right)\right] \\
=& \tilde{L}\left(\varepsilon_1 f_{t_1} + \sum_{k=2}^n \varepsilon_k \left(f_{t_k} - f_{t_{k-1}}\right)\right) \\
\leq& \|\tilde{L}\|_{(B([a,b]))^*} \cdot \left\|\varepsilon_1 f_{t_1} + \sum_{k=2}^n \varepsilon_k \left(f_{t_k} - f_{t_{k-1}}\right)\right\|_\infty .
\end{aligned}$$

Denote

$$F(\tau) = \varepsilon_1 f_{t_1}(\tau) + \sum_{k=2}^n \varepsilon_k \left[f_{t_k}(\tau) - f_{t_{k-1}}(\tau)\right],$$

then for each $\tau \in [a, b]$, only one of the terms f_{t_1} and $f_{t_k} - f_{t_{k-1}}$, $2 \leq k \leq n$, is nonzero, moreover, $|F(\tau)| = 1$, therefore $\|F\|_\infty = 1$, hence

$$\sum_{k=1}^n |g(t_k) - g(t_{k-1})| \leq \|\tilde{L}\|_{(B([a,b]))^*} = \|L\|_{(C([a,b]))^*},$$

for every partition of $[a, b]$, which implies $g \in BV([a, b])$ with $V_a^b(g) \leq \|L\|_{(C([a,b]))^*}$.

We claim that for the g defined above,

$$L(f) = \int_a^b f dg, \quad \forall f \in C([a,b]),$$

where $\int_a^b f dg$ is the Riemann-Stieltjes integral. Indeed, for arbitrary given $f \in C([a,b])$, and the partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

we define a function h_n as

$$f_n(\tau) = f(t_0) f_{t_1}(\tau) + \sum_{k=2}^{n} f(t_{k-1}) \left[f_{t_k}(\tau) - f_{t_{k-1}}(\tau) \right].$$

Then $f_n \in B([a,b])$. By the definition of g,

$$\begin{aligned}
\tilde{L}(h_n) =& f(t_0)\tilde{L}(f_{t_1}) + \sum_{k=2}^{n} f(t_{k-1}) \left[\tilde{L}(f_{t_k}) - \tilde{L}(f_{t_{k-1}}) \right] \\
=& f(t_0) g(t_1) + \sum_{k=2}^{n} f(t_{k-1}) \left[g(t_k) - g(t_{k-1}) \right] \\
=& \sum_{k=1}^{n} f(t_{k-1})[g(t_k) - g(t_{k-1})],
\end{aligned}$$

where we have used the fact $g(a) = 0$. The right-hand side of this chain of equalities is the Riemann-Stieltjes sum for the integral $\int_a^b f dg$. Therefore

$$\int_a^b f dg = \lim_{n\to\infty} \tilde{L}(f_n).$$

Since $f_n(a) = f(a)$ and for $t \in (t_{k-1}, t_k]$, $1 \le k \le n$

$$|f_n(t) - f(t)| = |f(t_{k-1}) - x(t)|,$$

then

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0,$$

therefore by the continuity of $\tilde{L}$,

$$\lim_{n\to\infty} \tilde{L}(f_n) = \tilde{L}(f),$$

which implies

$$\int_a^b f dg = \tilde{L}(f) = L(f),$$

where we have used the fact that $\tilde{L}$ is a extension of L. Moreover,

$$|L(f)| = \left| \int_a^b f dg \right| \le \max_{t\in[a,b]} |f(t)| \cdot V_a^b(g),$$

implies

$$\|L\|_{(C([a,b]))^*} \le V_a^b(g),$$

therefore $\|L\|_{(C([a,b]))^*} = V_a^b(g)$. □

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